Quantum Non-Demolition Test of Bipartite Complementarity

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We present a quantum circuit that implements a non-demolition measurement of complementary single- and bi-partite properties of a two-qubit system: entanglement and single-partite visibility and predictability. The system must be in a pure state with real coefficients in the computational basis, which allows a direct operational interpretation of those properties. The circuit can be realized in many systems of interest to quantum information.

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Quantum measurements frequently lead to a "back-action" on the observable being measured. This is the case for instance in the measurement of the position of a particle, which disturbs its momentum, thus affecting the future values of its position. This back-action can be overcome by using a quantum non-demolition (QND) scheme, introduced in [1]. In QND measurements, the observable $\mathcal{O}_S$ of a system S is measured by detecting a change in an observable $\mathcal{O}_P$ of a probe P coupled to S during a finite time, without perturbing the subsequent evolution of $\mathcal{O}_S$; after the measurement, the final state remains an eigenvector of $\mathcal{O}_S$. Experimental implementations have been performed in the optical domain, for measuring the intensity of an electromagnetic field [2] or the polarization of a photon [3], and in cavity quantum electrodynamics [4]. The characterization of QND measurements on qubit systems were discussed in Ref. [5].

Extension of the QND concept to bipartite systems poses quite a challenge, since entanglement measures, like the concurrence introduced by Wootters [6], do not have a direct operational meaning. Also, in the same way that the measurement of the photon number leads to complete uncertainty on the phase of the field [7], determination of the entanglement of a pair of qubits should lead to uncertainty on a complementary variable, and vice-versa. Identification of this complementary quantity is thus an important ingredient in understanding the QND scheme.

In this paper, we propose a quantum circuit for QND measurements of complementary variables in two-qubit systems described by pure states with real coefficients in the computational basis – named rebits in Ref. [8]. This restriction allows one to attach an operational meaning to those variables. One of them is the concurrence, while the other is a measure of the single-partite character of the global system. QND determination of the entanglement of a pair of qubits generates maximally entangled states even if the incoming state is a product state, analogous to the QND measurement of the number of photons in a cavity, which results in a Fock state [7]. It also leads to complete loss of single-partite properties. These are expressed as a sum of two contributions, standing for predictability and visibility, which in a double-slit Young interference correspond to the well-known duality between which-way information and the appearance of interference fringes. Bi-partite and single-partite properties are thus seen as complementary aspects, thus generalizing to bi-partite systems the concept of wave-particle duality.

The concept of complementarity is commonly related to mutually exclusive properties of single-partite quantum systems, the best known example being provided by the quantum interpretation of Young's double-slit experiment. Quantitative relations between visibility of interference fringes and distinguishability, corresponding to which-path information, were derived in [9, 10]. Quantification of the concept of complementarity for multipartite systems is a relatively recent undertaking. A complementarity relation between single- and two-particle visibility (which is an intrinsic bipartite property) was introduced in Refs. [11]. In [12] a possible connection between the distinguishability and an entanglement measure was hinted at, and in [11, 13] an intimate relation was established between concurrence [6] and the two-particle visibility in an interferometric setup. Prompted by these observations, in [14] it was shown that there is an underlying generalized complementarity relation of which these more restricted relations emerge as special cases. For two-qubit pure states, the general complementarity relation of [14] can be mathematically expressed as:

$$\mathcal{C}^2 + \mathcal{V}_k^2 + \mathcal{P}_k^2 = 1 \ , \tag{1}$$

where the ingredients are the concurrence $\mathcal{C}$, a genuine bipartite entity, and the single-partite visibility $\mathcal{V}_k$ and predictability $\mathcal{P}_k$ (for particle $k = 1, 2$). For mixed states, the sum of the three terms on the left-hand side of the above equation is smaller than one.

For a pure state $|\chi\rangle$, the visibility $\mathcal{V}_k$, a measure of the single-particle coherence (wave-like aspect), is defined as

$$\mathcal{V}_k = 2|\langle\chi|\,\sigma_k^+\,|\chi\rangle|, \text{ with } \sigma_k^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ . \tag{2}$$

Perfect visibility ($\mathcal{V}_k = 1$) is obtained for the states $(|0\rangle \pm \exp(i\phi)\,|1\rangle)/\sqrt{2}$.

The predictability $\mathcal{P}_k$, the particle-like aspect, a measure of the single-particle relative population, is defined for a pure state as:

$$\mathcal{P}_k = |\,\langle\chi|\,\sigma_k^z\,|\chi\rangle\,|, \text{ with } \sigma_k^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad (3)$$

Perfect predictability ($\mathcal{P}_k = 1$) is obtained for the eigenstates of σ_z, that is, the states $|0\rangle$ and $|1\rangle$.

Finally, for pure states the concurrence is defined as [6]

$$\mathcal{C} = |\,\langle\chi^*|\,\sigma_y \otimes \sigma_y\,|\chi\rangle\,|, \text{ with } \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \qquad (4)$$

One should note that the visibility and the predictability are not invariant under local (single-particle) unitary transformations, which can actually transform one into the other. For instance, a unitary transformation takes the maximum-predictability state $|0\rangle$ into the maximum-visibility state $(|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$. However, the quantity $\mathcal{S}_k^2 = \mathcal{V}_k^2 + \mathcal{P}_k^2$ is invariant under local unitary transformations, and can be considered as the proper measure for the single-partitedness of the global system. With this definition, one can read Eq. (1) as a *duality* relation between bipartite and single partite properties,

$$\mathcal{C}^2 + \mathcal{S}_k^2 = 1\ . \qquad (5)$$

One may say therefore that the single-partite property and the bipartite property of a two-particle state are complementary just as the wave and particle properties of single-particle systems are complementary. While visibility and predictability are properties of an individual particle, and exhaust for a single-particle system the full content of wave-particle duality, for a bipartite system the concurrence, a genuine bipartite quantity, also enters into the complementarity relation.

We now discuss in detail our method for implementing a QND measurement of the complementary quantities in Eq. (1). To this end we first note that a general bipartite pure qubit state can be written in the Bell basis as

$$|\chi\rangle = \alpha\,|\psi^-\rangle + \beta\,|\psi^+\rangle + \gamma\,|\phi^-\rangle + \eta\,|\phi^+\rangle\ , \qquad (6)$$

where $|\psi^\pm\rangle = (|10\rangle \pm |01\rangle)/\sqrt{2}$, $|\phi^\pm\rangle = (|11\rangle \pm |00\rangle)/\sqrt{2}$ are the Bell states and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\eta|^2 = 1$.

For this state one has:

$$\begin{aligned} \mathcal{V}_{\frac{1}{2}} &= 2|\Re(\beta^*\eta \mp \alpha^*\gamma) + i\Im(\beta\gamma^* \pm \eta\alpha^*)|; & (7)\\ \mathcal{P}_{\frac{1}{2}} &= 2|\Re(\alpha^*\beta \pm \eta^*\gamma)|; & (8)\\ \mathcal{C} &= |\alpha^2 - \beta^2 - \gamma^2 + \eta^2|. & (9) \end{aligned}$$

The definition of concurrence involves state conjugation, a non-physical operation, and therefore this quantity cannot be directly measured in the general case. For pure states, direct detection of entanglement has been demonstrated by making a measurement on two copies of a state [15]. If one measures just one copy at a time, however, one must further specialize the state in order for concurrence to be given an operational meaning. Equation (4) implies that concurrence is the magnitude of the average of ßfor all states with real coefficients in the computational or Bell-state basis. Therefore, it can be given an operational meaning for this class of states, thus providing the possibility of directly measuring each term in Eq. (1). For this reason, from now on we will be dealing only with the case of real coefficients. Even though this limits the general applicability of the method, one should note that real quantum computation has the full quantum computation power as was shown in Ref. [16].

For this class of states, the visibility is given by $|\langle\chi|\sigma_x|\chi\rangle|$. Therefore, the quantities in Eq. (1) can be expressed in terms of averages of the operators (taking $k = 1$ for definiteness) $\hat{V}_1 = \sigma_x \otimes \mathbb{1}$, $\hat{P}_1 = \sigma_z \otimes \mathbb{1}$, and $\hat{C} = \sigma_y \otimes \sigma_y$. Since these operators do not commute, a QND measurement of one of them would necessarily spoil the determination of the other. Thus, for instance, the QND measurement of $\hat{C}$ leads to an eigenstate of this observable, with eigenvalue ± 1, yielding a state with maximal concurrence (equal to one), which is not an eigenstate of $\hat{V}_1$ or $\hat{P}_1$. In fact, the averages of these operators in the resulting state are equal to zero, thus yielding zero visibility and predictability, as expected from Eq. (1). The uncertainty relation among these three observables is better expressed in terms of the sum of variances: $(\Delta\hat{V}_1)^2 + (\Delta\hat{P}_1)^2 + (\Delta\hat{C})^2 = 2$, where $(\Delta\mathcal{O})^2 = \langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2$. Since each variance is at most one, this relation shows that when one of the observables is perfectly known, the two others must have maximum variance. The uncertainty relations involving products of variances are not useful in this case, since for instance $\Delta\hat{V}_1\Delta\hat{P}_1 = |\langle\sigma_y \otimes \mathbb{1}\rangle|/4$, and the right-hand side vanishes for an eigenstate of $\hat{P}_1$, so that $\Delta\hat{V}_1$ is undetermined.

We show now that there is a general circuit, involving three adjustable parameters, which implements QND measurements of these three observables. We start with a simpler scheme that measures the concurrence, and then consider a more general scheme, which performs a QND measurement of all three quantities in Eq. (1).

QND measurement of concurrence. The corresponding circuit is shown in Fig. 1. It consists of single-qubit rotations and controlled-not (CNOT) gates, which are the fundamental building blocks of QND measurements.

The composite state, given initially by Eq. (6) evolves as follows. The gates $R_x(\pi/2)$ transform it into:

$$|\chi\rangle\,|0\rangle \rightarrow (\alpha\,|\psi^-\rangle - i\eta\,|\psi^+\rangle + \gamma\,|\phi^-\rangle - i\beta\,|\phi^+\rangle)\,|0\rangle\ . \qquad (10)$$

The two CNOT gates lead to the state:

$$(\alpha\,|\psi^-\rangle - i\eta\,|\psi^+\rangle)\,|1\rangle + (\gamma\,|\phi^-\rangle - i\beta\,|\phi^+\rangle)\,|0\rangle\ . \qquad (11)$$

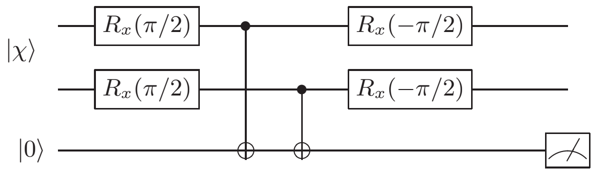

FIG. 1: Quantum circuit for QND measurement of concurrence. $|\chi\rangle$ is the two-qubit input state, the ancilla qubit is initially in the state $|0\rangle$ state, and $R_x(\pi/2) = \exp(-i\pi\sigma_x/4)$.

After the final rotations one has:

$$\left(\alpha \left|\psi^-\right\rangle + \eta \left|\phi^+\right\rangle\right)|1\rangle + \left(\gamma \left|\phi^-\right\rangle + \beta \left|\psi^+\right\rangle\right)|0\rangle \ . \tag{12}$$

The final step is to perform the measurement on the ancilla state. Thus the conditional outgoing states are:

$$\begin{cases} |\chi_1\rangle = \frac{\alpha|\psi^-\rangle+\eta|\phi^+\rangle}{\sqrt{\alpha^2+\eta^2}} \text{ if the ancilla is in } |1\rangle \\ \\ |\chi_0\rangle = \frac{\gamma|\phi^-\rangle+\beta|\psi^+\rangle}{\sqrt{\beta^2+\gamma^2}} \text{ if the ancilla is in } |0\rangle \end{cases} . \tag{13}$$

The concurrence of these states is easily calculated to be:

$$\begin{cases} \mathcal{C}(\chi_1) \ = \ |\alpha^2+\eta^2|/(\alpha^2+\eta^2) = 1 \\ \mathcal{C}(\chi_0) \ = \ |-\beta^2-\gamma^2|/(\beta^2+\gamma^2) = 1 \end{cases} . \tag{14}$$

The concurrence of the outgoing state is equal to 1 independent of the result of the ancilla measurement, provided the coefficients in the initial state are real. Thus, the outgoing state is maximally entangled for any input state. We can even start with a separable state such as $|00\rangle$, for example, and the final state will still be either $|\phi^+\rangle$ or $|\phi^-\rangle$, depending on the ancilla measurement outcome. The concurrence of the initial state is determined from the statistics of the measurements on the ancilla for many realizations of the experiment:

$$|p_1 - p_0| = |\alpha^2 - \beta^2 - \gamma^2 + \eta^2| = C(\chi)\,; \tag{15}$$

p_i being the probability of finding the ancilla in state i.

Although entanglement is invariant under local transformations, we undo the rotations in order to end up in a ßeigenvector, avoiding then the back action. The circuit thus measures in a QND way the expectation value $\langle\sigma_y \otimes \sigma_y\rangle$, the magnitude of which is the concurrence for real states.

QND measurement of single- and bi-partite features. In order to perform QND measurements of all the observables corresponding to the quantities in the complementarity relation (1), we need a circuit that allows the measurement of single-particle features as well. Such a circuit is presented in Fig. 2.

The previous result can be obtained by setting $\vec{\theta_3} = (\pi/2)\hat{y}$, so that the ancilla is prepared in a maximally entangled state $|\phi^+\rangle$. Only $|\phi^+\rangle$ and $|\psi^+\rangle$ are used, and these states act as a logical qubit. This circuit is

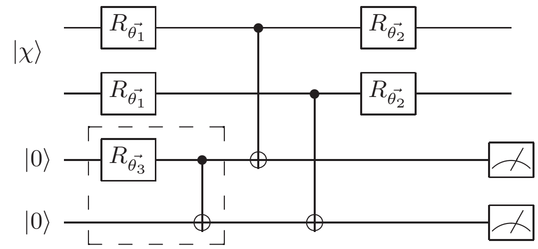

FIG. 2: Universal quantum circuit for QND measurement of concurrence, visibility and predictability. The dashed box is the ancilla state preparation and $R_{\vec{\theta_i}} = \exp(-i\vec{\sigma}\cdot\vec{\theta_i})$.

then completely equivalent to the one in Fig. 1, replacing $|0\rangle \to |\phi^+\rangle$ and $|1\rangle \to |\psi^+\rangle$. Here $\vec{\theta_1} = (\pi/2)\hat{x} = -\vec{\theta_2}$, as in the previous circuit. A local measurement in the computational basis distinguishes between $|\phi^+\rangle$ and $|\psi^+\rangle$. The probabilities are easily related: $p_{\phi^+} = p_{00} + p_{11}$ and $p_{\psi^+} = p_{10} + p_{01}$. The concurrence is now given by $|p_{\psi^+} - p_{\phi^+}|$.

For the QND measurement of $\hat{V}_k$ and $\hat{P}_k$, corresponding to visibility and predictability, we choose $\vec{\theta_3} = 0$, which leads to a separable ancilla state $|00\rangle$. In this case the odd lines of the circuit are completely decoupled from the even ones, thus yielding two independent circuits, which is a natural choice if one wants to measure single-particle aspects.

A single CNOT gate, without any rotation, would project the final state onto an eigenvector of σ_z, that is, onto one of the computational-basis states. The average of the measurements on the ancilla yield $\langle\sigma_z\rangle$, and hence this is a non-demolition measurement of the predictability. Therefore, measurement of the predictability corresponds to the choice $\vec{\theta_1} = \vec{\theta_2} = 0$.

The state before the measurement of the ancilla is then:

$$\frac{1}{\sqrt{2}}[(\eta-\gamma)\,|00\rangle\,|00\rangle + (\beta-\alpha)\,|01\rangle\,|01\rangle + \\ (\alpha+\beta)\,|10\rangle\,|10\rangle + (\gamma+\eta)\,|11\rangle\,|11\rangle]. \tag{16}$$

A measurement on the ancilla leads to an outgoing state with perfect predictability for both qubits. The probabilities for the several possible outcomes yield the predictabilities of the initial real state:

$$\begin{cases} |(p_{00}+p_{01}) - (p_{10}+p_{11})| = 2|\alpha\beta+\eta\gamma| = \mathcal{P}_1 \\ |(p_{00}+p_{10}) - (p_{01}+p_{11})| = 2|\alpha\beta-\eta\gamma| = \mathcal{P}_2 \end{cases} . \tag{17}$$

The first line represents the difference between the probabilities of measuring 0 and 1 for the first ancilla, while the second line is the difference between the probabilities of finding the second ancilla in either 0 or 1.

For the QND measurement of the visibility, one must perform a $\pi/2$ rotation around the $\hat{y}$ axis in state space, since the visibility for real states is related to the σ_x matrix. However, the visibility does change under local rotations, therefore the initial rotation must be undone

at the end of the circuit, in order to end up in a maximum visibility state for both qubits. Thus, one must have $\vec{\theta}_1 = (\pi/2)\hat{y} = -\vec{\theta}_2$, which leads to the following composite state right before the ancilla measurement:

$$\frac{1}{\sqrt{2}}[(\eta+\beta)\left|+\right\rangle\left|+\right\rangle\left|00\right\rangle + (\gamma-\alpha)\left|+\right\rangle\left|-\right\rangle\left|01\right\rangle \; + \\ (\alpha+\gamma)\left|-\right\rangle\left|+\right\rangle\left|10\right\rangle + (\eta-\beta)\left|-\right\rangle\left|-\right\rangle\left|11\right\rangle] \quad , \quad (18)$$

where $|\pm\rangle \equiv (|1\rangle \pm |0\rangle)/\sqrt{2}$.

The ancilla measurement in the computational basis will project the outgoing state onto a maximum visibility state. From the measurement statistics one can infer the initial-state visibility for both qubits:

$$\begin{cases} |(p_{00}+p_{01})-(p_{10}+p_{11})| = 2|\alpha\gamma-\eta\beta| = \mathcal{V}_1 \\ |(p_{00}+p_{10})-(p_{01}+p_{11})| = 2|\alpha\gamma+\eta\beta| = \mathcal{V}_2 \end{cases} . \quad (19)$$

With these two measurements one has a full QND characterization of the single-particle features. The outgoing state in both cases is separable.

It is easy to check that the above measurement scheme fulfills all the requirements for qubit QND measurements listed in Ref. [5]. The outgoing state is, after measurement, an eigenvector of the measured observable. For instance, when measuring concurrence the state of the system becomes a ßeigenvector. Also, $\mathcal{S}_k^2$ and the concurrence do not change in time due to free local evolution. The requirements for QND measurements are then fulfilled for both parts of the complementarity relation in Eq. (5). On the other hand, visibility and predictability can be interchanged between each other depending on the free Hamiltonian. For many cases of interest, however, the free Hamiltonian is proportional to $(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z)$ and, in these cases, the visibility and predictability measurements are themselves QND-like.

The above circuits can be implemented in many systems of interest for quantum information, since they involve single-particle rotations and CNOT gates, which have been demonstrated for instance in trapped ions [17], cavity QED [18] and with two pairs of twin photons, created as shown in Ref. [19].

In conclusion, we have shown that it is possible to implement independent QND measurements of all the complementary quantities corresponding to a two-qubit state, which express its single- and bipartite content. The restriction to states with real coefficients in the computational basis seems to be unavoidable in the present context, since otherwise it is not possible to attribute an operational meaning to concurrence for measurements realized on single copies of an ensemble.

These measurements illustrate the complementarity among single- and bipartite quantities: a QND measurement of entanglement leads to a maximally entangled state, but spoils at the same time the visibility and the predictability for each qubit. This could have broad implications for quantum information processing, since after determining the single-partite or bi-partite content of a quantum state, the state itself can be further processed; elimination of back-action guarantees that the measured value is preserved. Thus, after a measurement of entanglement, the resulting state could be used as a resource in, e.g., teleportation [20] and quantum cryptography [21] protocols.

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[1] V. Braginsky, Y. Vorontsov, and F. Khalili, Zh. Eks. Teor. Fiz. **73**, 1340 (1977).
[2] M. Levenson *et al.*, Phys. Rev. Lett. **57**, 2473 (1986). N. Imoto, S. Watkins, and Y. Sasaki, Opt. Commun. **61**, 159 (1987). A. LaPorta, R.E. Slusher, and B. Yurke, Phys. Rev. Lett. **62**, 28 (1989).
[3] G. J. Pryde *et al.*, Phys. Rev. Lett. **92**, 190402 (2004).
[4] G. Nogues *et al.*, Nature **400**, 239 (1999).
[5] T. C. Ralph *et al.*, Phys. Rev. A **73**, 012113 (2006).
[6] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[7] M. Brune *et al.*, Phys. Rev. A **45**, 5193 (1992).
[8] C. M. Caves, C. A. Fuchs, and P. Rungta, Found. Phys. Lett. **14**, 199 (2001).
[9] L. Mandel, Opt. Lett. **16**, 1882 (1991).
[10] B.-G. Englert, Phys. Rev. Lett. **77**, 2154 (1996).
[11] G. Jaeger, M.A. Horne, and A. Shimony, Phys. Rev. A **48**, 1023 (1993). G. Jaeger, A. Shimony, and L. Vaidman, Phys. Rev. A **51**, 54 (1995).
[12] B.-G. Englert and J. Bergou, Opt. Commun. **179**, 337 (2000).
[13] A. F. Abouraddy *et al.*, Phys. Rev. A **64**, 050101(R) (2001). M. Jakob and J. Bergou, Phys. Rev. A **66**, 062107 (2002).
[14] M. Jakob and J. A. Bergou, quant-ph/0302075 (2003).
[15] S.P. Walborn *et al.*, Nature **440**, 1022 (2006).
[16] E. Bernstein and U. Vazirani, Siam J. of Comp. **26**, 1411 (1997). Y. Shi, quant-ph/0205115.
[17] F. Schmidt-Kaler *et al.*, Nature **422**, 408 (2003); D. Leibfried *et al.*, Nature **422**, 412 (2003).
[18] A. Rauschenbeutel *et al.*, Phys. Rev. Lett. **83**, 005166 (1999).
[19] S. Gasparoni *et al.* Phys. Rev. Lett. **93**, 020504 (2004).
[20] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993). J.-W. Pan *et al.*, Nature **421**, 721 (2003).
[21] C. H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992).